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# Min(Max-PSD) and Max(Min-PSD) as lifetime distributions in Network's Reliability

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*Dedicated to the memory of Professor Constantin Popa*

**Abstract**

In our paper we describe the corresponding dynamic mathematical models to perform a comparative analysis of the reliability of two types of networks: serial-parallel and parallel-serial when the number of sub-networks is constant and the numbers of units in each sub-network are Power Series Distributed (PSD) random variables (r.v.), but also when the lifetimes are independent, identically distributed r.v. We shows that the lifetime distributions of such kind of networks leads us to two new families of distributions called Min(Max-PSD) and Max(Min-PSD) distributions. The formulas for calculating the reliability of the related networks it was deduced too. Sufficient conditions have been formulated for the serial-parallel network to always be more reliable than the parallel-serial network. Some graphically illustrated examples have been provided.

## 1. Introduction

Key Words: lifetime distributions, PSD distributions, survival function / reliability, serial-parallel and parallel-serial networks.
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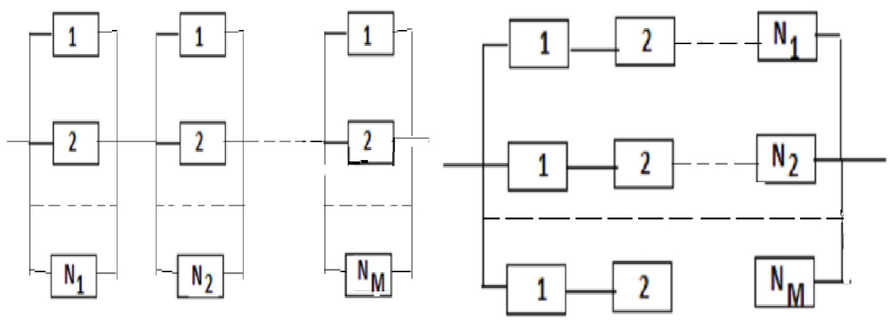
First of all, let us observe that many mathematical models in Network's Reliability deal with series and parallel Networks as subsystems of Networks with more complex structure/topology. If for elementary networks, such as serial or parallel networks, it is obvious that the latter are, under the stated conditions, more reliable, then in the case of more complex networks, consisting of some and the same elements, the answer is not so obvious.

Let us take, for beginning, two standard network types: serial-parallel ($A$) and parallel-serial ($B$), according to the schemes below.




$A$. Serial-Parallel Network. $B$. Parallel-Serial Network .

Next we will adopt dynamic mathematical models for the networks represented above. More exactly, we suppose the following: the lifetime of each network's unit is described as a nonnegative r.v. whose cumulative distribution function (c.d.f.) is known and lifetimes of such units are independent, identically distributed random variables (i.i.d.r.v.). Reliability of the network will be represented by the survival function, i.e., the probability that the network will survive a longer time than $x$, which coincides, in fact, with the tail of c.d.f. for the lifetime of the entire network.

In the following we are interested in: 1. Obtaining calculation formulas for the lifetime distributions and reliability of networks of type $A$ or $B$ in different conditions; 2. Comparative analysis of the networks invoked from the point of view of their reliability.

## 2. Notions and auxiliary results

To include as possible many mathematical models in one formula, we will use, as generic distribution of the number of units in each subnetwork, the

$0-$truncated Power Series Distributions (PSD) class obtained by means of the $0-$truncation of each power series distribution. The term "power series distributions" is generally credited to Noack [6]. Kosamby [7] and Noack showed that PSD class include Binomial, Poisson, Logarithmic, Geometric, Negative Binomial, Pascal and many other important discrete distributions. More exactly we have

**Definition 1** [6]**.** We say that *$Z$ is a Power Series Distributed r.v. with parameter $\theta$ and power series function $A(\theta) = \sum_{z\geqslant 0} a_z\theta^z$ , shortly $Z \in PSD$,* if

$$\mathbf{P}(Z = z) = \frac{a_z\theta^z}{A(\theta)},\ a_z \geqslant 0,\ z = 0, 1, 2, ...; \theta \in [0, \tau),$$

where power series $\sum_{z\geqslant 0} a_z\theta^z$ is convergent with radius of convergence $\tau \in (0, +\infty)$.

The PSD discrete probability distributions used in our paper are $0-$truncated ones, because the real networks consists from at least one unit, The following assertion assure us that the operation of $0-$truncations does not alter the initial quality of distribution to be of PSD class..

**Proposition 1** $[3, 4]$**.** *If r.v. $Z \in PSD$ with parameter $\theta \in (0, \tau)$, $\tau \in (0, +\infty)$ and power series function $A(\theta) = \sum_{z\geqslant 0} a_z\theta^z$, then his 0-truncation is a r.v. $Z^* \in PSD$ with parameter $\theta \in [0, \tau)$, $\tau \in (0, +\infty)$ and power series function $A^*(\theta) = \sum_{z\geqslant 1} a_z\theta^k = A(\theta) - a_0$, i.e.,*

$$\mathbf{P}(Z^* = z) = \frac{a_z\theta^z}{A^*(\theta)} = \frac{a_z\theta^z}{A(\theta) - a_0},\ a_z \geqslant 0,\ z = 1, 2, ... .$$

**Remark 1.** *If r.v. $Z \in PSD$* and the null coefficient $a_0$ of his power series function $A(\theta) = \sum_{z\geqslant 0} a_z\theta^z$ is equal to 0, then 0-truncation of r.v. $Z$ does not change the initial distribution.

**Example 1.** The following Table 1 shows the form of PSD parameters of two distributions used by us to exemplify our theoretical results, i.e., $Bin(n; p)$ and $Pascal(k; p)$, marked by symbol " $*$ " , if their 0-truncation changes form as a PSD. Here, $Bin(n; p)$ expresses the probability distribution of total number of "success" occurring in $n$ Bernoulli trials with the same probability $p = P($"*succes*"$)$ in each trial. At the same time, $Pascal(k; p)$ expresses the probability distribution of the number of Bernoulli trials until $k$ "successes" have occurred when the probability of a "success" in each trial is $p$.

| Distribution | $a_z$ | $\theta$ | $A(\theta)$ | $\tau$ |
|---|---|---|---|---|
| $Bin^*(n;p)$, $n \in \{1,2,..\}$, $0<p<1$ | $\begin{cases} \binom{n}{z}, \textit{for } z = \overline{1,n}, \\ 0, \textit{ for } z=0 \textit{ or } z>n. \end{cases}$ | $\frac{p}{1-p}$ | $(1+\theta)^n - 1$ | $+\infty$ |
| $Pascal(k;p)$, $k \in \{1,2,..\}$, $0<p<1$ | $\begin{cases} \binom{z-1}{k-1}, \textit{for } z = k, k+1, .., \\ 0, \textit{ for } z = \overline{0,k-1}. \end{cases}$ | $1-p$ | $\left(\frac{\theta}{1-\theta}\right)^k$ | 1 |

*Table* 1.

Next results refers to the distributions of PSD mixtures of minimum or maximum of nonnegative i.i.d.r.v.

**Proposition 2** [2,4]. *If $X_1$, $X_2$,...,$X_n$,... are nonnegative i.i.d.r.v. with c.d.f. $F(x) = \mathbf{P}(X_i \le x)$, $i \ge 1$ and r.v. $N \in PSD$ with parameter $\theta \in (0,\tau)$, $\tau \in (0,+\infty)$ and with power series function $A(\theta) = \sum\limits_{k \geqslant 1} a_k \theta^k$, $N$ being independent of r.v. $X_1$, $X_2$,...,$X_n$,..., then c.d.f. of r.v. $U_N = \min(X_1, X_2,...,X_N)$ and $V_N = \max(X_1, X_2,...,X_N)$ are given, respectively, by formulas*

$$F_{U_N}(x) = \mathbf{P}(U_N \le x) = 1 - \frac{A(\theta(1-F(x))}{A(\theta)},$$

$$F_{V_N}(x) = \mathbf{P}(V_N \le x) = \frac{A(\theta F(x))}{A(\theta)}.$$

According to the papers [2,5], the above lifetime distributions will be called *lifetime distributions of Min-PSD type* and, respectively of *Max-PSD type*.

**Remark 2.** The c.d.f. $F_{U_N}(x)$ and $F_{V_N}(x)$ describe probabilistic behavior of lifetimes for serial and parallel networks, respectively, when lifetimes $X_1$, $X_2$,...,$X_n$,...of the units are nonnegative i.i.d.r.v. and number of the units is a r.v. $N \in PSD$, $N$ being independent of r.v. $X_1$, $X_2$,...,$X_n$,... .

## 3. Reliability of serial-parallel and parallel-serial networks with constant number of subnetworks and random number of units in each subnetwork.

Now, let us suppose that Networks $A$ and $B$ consist from constant number $M > 1$ of subnetworks, where the numbers $N_1$, $N_2$, ..., $N_M$ of units in the corresponding subnetworks are independently, identically, Power Series Distributed random variables with parameter $\theta \in (0,\tau)$, $\tau \in (0,+\infty)$ *and with* *power series function* $A(\theta) = \sum_{k\geqslant 1} a_k\theta^k$ and, at the same time, independents of lifetimes of all units. Lifetimes of all units compounding the Network being non-negatives, i.i.d.r.v. with c.d.f. $F(x)$, then lifetimes of all $M$ subnetworks in the serial-parallel network are i.i.d.r.v. $V_{N_1}$, $V_{N_2}$,..., $V_{N_M}$. In the same way, lifetimes of all $M$ subnetworks in the parallel-serial network are i.i.d.r.v. $U_{N_1}$, $U_{N_2}$,..., $U_{N_M}$. More than,

$$U_{N_i} = \min(X_1,..., X_{N_i}) \text{ and } V_{N_i} = \max(X_1,..., X_{N_i}),\ i = \overline{1, M}.$$

So, according to the Proposition 2, their c.d.f. as a Min-PSD and Max-PSD distributions may be calculated by the following formulas:

$$F_{U_{N_i}}(x) = \mathbf{P}(U_{N_i} \leq x) = 1 - \frac{A\left(\theta(1 - F(x))\right)}{A(\theta)},$$

$$F_{V_{N_i}}(x) = \mathbf{P}(V_{N_i} \leq x) = \frac{A(\theta F(x))}{A(\theta)},\ i = \overline{1, M}.$$

Because lifetimes for serial-parallel and parallel-serial networks correspond, respectively, to the r.v. $U_{S-P}$=min($V_{N_1}$,$V_{N_2}$,...,$V_{N_M}$), $V_{P-S}$=max($U_{N_1}$, $U_{N_2}$,..., $U_{N_M}$), we deduce that their c.d.f. may be calculated by means of the formulas:

$$F_{U_{S-P}}(x) = \mathbf{P}(U_{S-P} \leq x) = 1 - \left(1 - \frac{A(\theta F(x))}{A(\theta)}\right)^M, \quad \mathbf{(1)}$$

$$F_{V_{P-S}}(x) = \mathbf{P}(V_{P-S} \leq x) = \left(1 - \frac{A(\theta(1 - F(x)))}{A(\theta)}\right)^M, \ \mathbf{(2)}.$$

Due to the structure of the lifetimes $U_{S-P}$ and $V_{S-P}$, it is natural to call the distribution (1) -distribution of Min(Max-PSD) type and the distribution (2) -distribution of Max(Min-PSD) type.

At the same time, in the same conditions, the corresponding survival/reliability functions are given by the following formulas:

$$S_{S-P}(x) = \mathbf{P}(U_{S-P} > x) = \left(1 - \frac{A(\theta F(x))}{A(\theta)}\right)^M, \quad \mathbf{(3)}$$

$$S_{P-S}(x) = \mathbf{P}(V_{P-S} > x) = 1 - \left(1 - \frac{A(\theta(1 - F(x))}{A(\theta)}\right)^M, \ \mathbf{(4)}.$$

Due to the characteristic properties of c.d.f. $F(x)$, we have that $F(x)$ is monotonous non-decreasing function and $0 \leq F(x) \leq 1$, for every $x \in (-\infty, +\infty)$. So, if we denote by $q = F(x)$ for fixed $x$, then, according to the formulas **(3)-(4)**, the comparison of the corresponding survival/reliability functions will be equivalent to the comparison of the functions $S_{S-P}(q) = \left(1 - \frac{A(\theta q)}{A(\theta)}\right)^M$ and $S_{P-S}(q) = \left[1 - \left(1 - \frac{A(\theta(1-q))}{A(\theta)}\right)^M\right]$ for $0 \leqslant q \leqslant 1$, independent of how c.d.f. $F(x)$ is looking like. In other words, we may consider that the lifetime of each unit is described by uniform distribution on the interval $[0, 1]$ , i.e., the c.d.f.

$$F(x) = \begin{cases} 0, \ x < 0, \\ x, \ 0 \leq x \leq 1, \\ 1, \ x > 1. \end{cases}$$

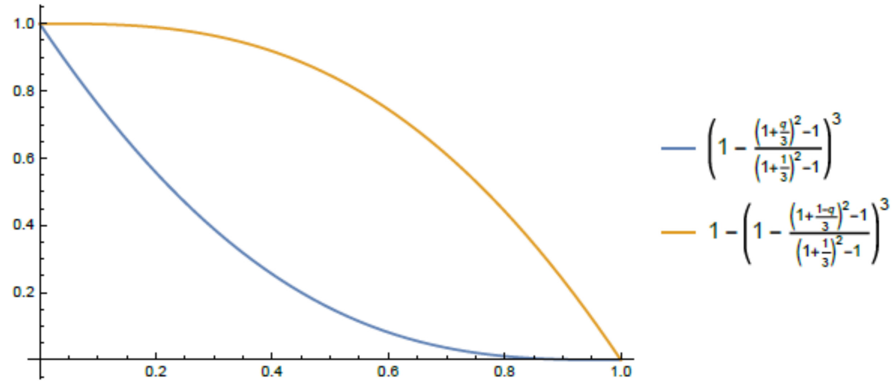
**Remark 3.** In fact, the previous conclusion means that the choice of the most reliable network does not depend of lifetime distribution $F(x)$ and may depend only of the probability distribution of the number of units in each of $M$ subnetworks. The following example confirms our assertion.

**Example 2.** Let us consider serial-parallel and parallel-serial networks with constant number $M$ of subnetworks and numbers $N_1$, $N_2$,..., $N_M$ of the units in the corresponding subnetworks as the independent, identically, $0-$truncated, binomially distributed r.v. with parameters $n$ and $p \in (0, 1)$. From the Table 1 we see that this is a PSD with power function $A(\theta) = (1+\theta)^n - 1$, $0 < \theta < +\infty$, where $\theta = \frac{p}{1-p}$. So,

$$S_{S-P}(q) = \left[1 - \frac{[1+\theta q]^n - 1}{[1+\theta]^n - 1}\right]^M, \ S_{P-S}(q) = 1 - \left[1 - \frac{[1+\theta(1-q)]^n - 1}{[1+\theta]^n - 1}\right]^M.$$

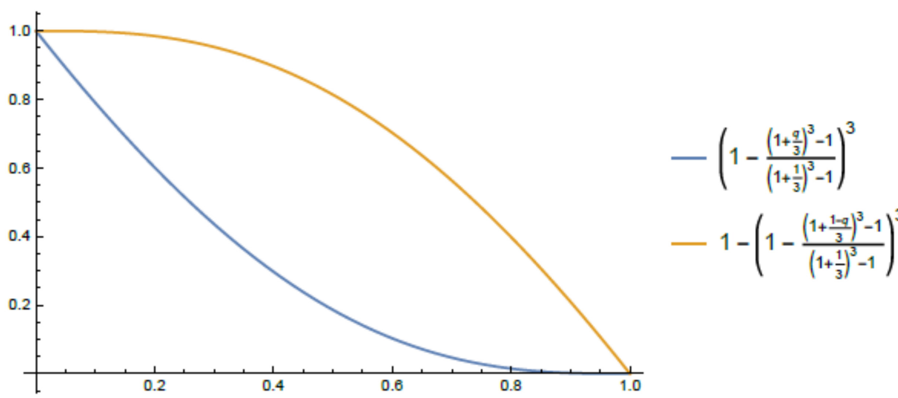
Then the below graphical representations of $S_{S-P}(q)$ and $S_{P-S}(q)$ show us that the comparison of the reliability of networks $A$ and $B$ is not predetermined.

*a*) for $M = 3$, $n = 2$, $p = 1/4$, $\theta = \frac{p}{1-p} = \frac{1}{3}$ we have that the functions $S_{S-P}(q) = \left[1 - \frac{[1+\frac{1}{3}q]^2-1}{[1+\frac{1}{3}]^2-1}\right]^3$, $S_{P-S}(q) = 1 - \left[1 - \frac{[1+\frac{1}{3}(1-q)]^2-1}{[1+\frac{1}{3}]^2-1}\right]^3$ and their graphical representation is the following:



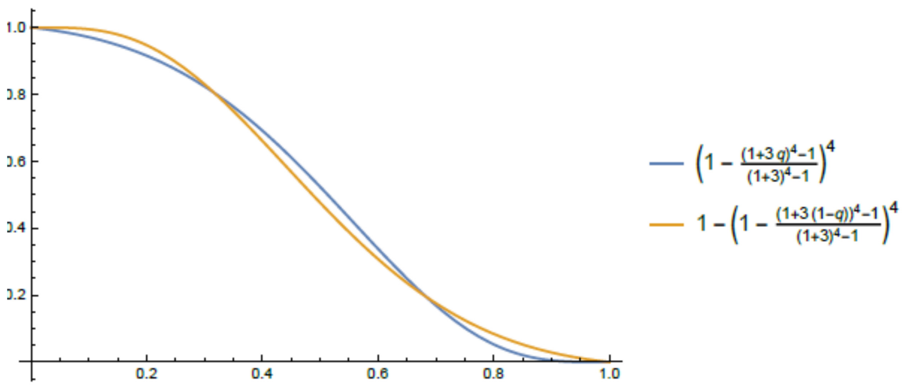

The situation does not change even $n = M = 3$.

*b*) for $M = 3$, $n = 3$, $p = 1/4$, $\theta = \frac{p}{1-p} = \frac{1}{3}$ we have that $g_{S-P}(q) = \left[1 - \frac{[1+\frac{1}{3}q]^3-1}{[1+\frac{1}{3}]^3-1}\right]^3$, $g_{P-S}(q) = 1 - \left[1 - \frac{[1+\frac{1}{3}(1-q)]^3-1}{[1+\frac{1}{3}]^3-1}\right]^3$ and their graphical representation is the following:



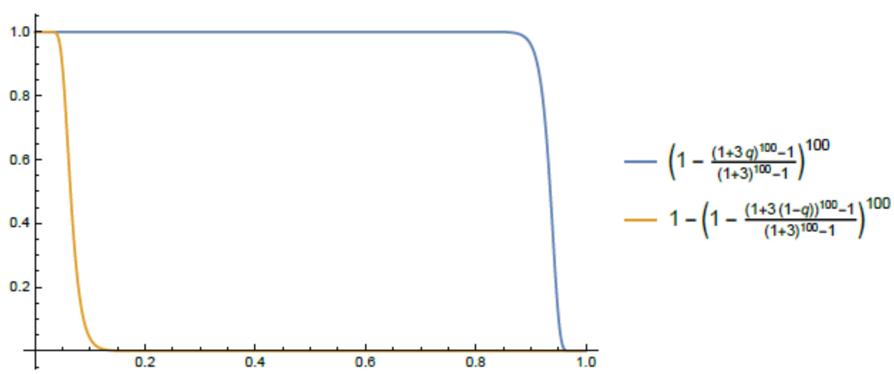

But the situation when $M = n$ changes dramatically if $M = n \geq 4$. See, for example, the following case:

*c*) for $M = 4$, $n = 4$, $p = 3/4$, $\theta = \frac{p}{1-p} = 3$ we have that the functions $S_{S-P}(q) = \left[1 - \frac{[1+3q]^4-1}{[1+3]^4-1}\right]^4$, $S_{P-S}(q) = 1 - \left[1 - \frac{[1+3(1-q)]^4-1}{[1+3]^4-1}\right]^4$ and their graphical representation is the following:



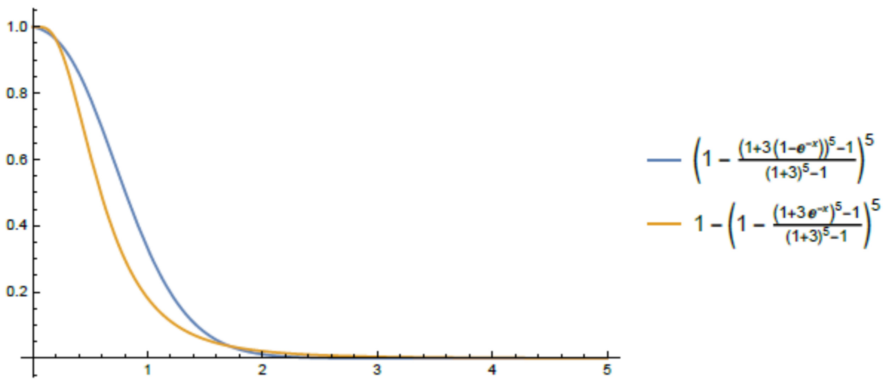

The following case show that with the increase of $M$, gradually, in the case of $M = n$, the series-parallel structures becomes more and more reliable than the parallel-series structures:

*d*) for $M = 100$, $n = 100$, $p = 3/4$, $\theta = \frac{p}{1-p} = 3$ we have that the functions $S_{S-P}(q) = \left[1 - \frac{[1+3q]^{100}-1}{[1+3]^{100}-1}\right]^{100}$, $S_{P-S}(q) = 1 - \left[1 - \frac{[1+3(1-q)]^{100}-1}{[1+3]^{100}-1}\right]^{100}$ and their graphical representation is the following:




At the same time, if we use another c.d.f., for example, $F(x) = 1-e^{-x}$, $x \geq 0$, instead of $q \in [0, 1]$, i.e., instead of uniform distribution on the interval $[0, 1]$, then the graphical representations of survival/reliability functions $S_{S-P}(x)$ and $S_{P-S}(x)$ confirms our Remark 3. Let us take, for example, analog of case *c*).

*e*) for $M = 5$, $n = 5$, $p = 3/4$, $\theta = \frac{p}{1-p} = 3$ we have that the functions $S_{S-P}(q) = \left[1 - \frac{[1+3(1-e^{-x})]^5 - 1}{[1+3]^5 - 1}\right]^5$, $S_{P-S}(q) = 1 - \left[1 - \frac{[1+3e^{-x}]^5 - 1}{[1+3]^5 - 1}\right]^5$ and their graphical representation is the following:




**Remark 4.** All of the above examples refer, in fact, to the case when the probability, that the number of units in each sub-network is greater than the number of sub-networks $M$, is less than 1. They show that in this case the comparison of the reliability of networks $A$ and $B$ is not predetermined.

On the other hands, in the case when numbers of units $N_1$, $N_2$, ..., $N_M$ and number of subnetworks $M$ are constant numbers it was proved the following

**Proposition 3** [3]**.** *For $M > 1$, if the lifetimes of the network units are i.i.d.r.v. and $\min(N_1, N_2, ..., N_M) > M$, then the survival / reliability functions $S_{S-P}(x) > S_{P-S}(x)$, i.e., networks of type $A$ are more reliable than networks of type $B$, regardless of the lifetime distribution of their units.*

That means in our case we have the following

**Consequence.** *For $M > 1$, if the lifetimes of the network units are i.i.d.r.v. and $N_1, N_2, ..., N_M$ are i.i.d.r.v. of PSD type such that $P(N_i > M) = 1$ for every $i = \overline{1, M}$, then the survival / reliability functions $S_{S-P}(x) > S_{P-S}(x)$, i.e., networks of type $A$ are more reliable than networks of type $B$, regardless of the lifetime distribution of their units.*

**Example 3.** Let us consider serial-parallel and parallel-serial networks with constant number $M$ of subnetworks and numbers $N_1$, $N_2$,..., $N_M$ of the units in the corresponding subnetworks as the independent, identically, $0-$truncated, Pascal distributed r.v. with parameters $k > M$ and $p \in (0, 1)$. So, $P(N_i > M) = 1$, because $P(N_i \in \{k, k+1, ..\}) = 1$, where $k > M$. From the Table 1 we see that this is a PSD with power function $A(\theta) = (\frac{\theta}{1-\theta})^k$,

$0 < \theta < 1$, where $\theta = 1 - p$. So,

$$S_{S-P}(q) = \left[1 - \frac{(\frac{\theta q}{1-\theta q})^k}{(\frac{\theta}{1-\theta})^k}\right]^M = \left[1 - \frac{[q(1-\theta)]^k}{(1-\theta q)^k}\right]^M,$$

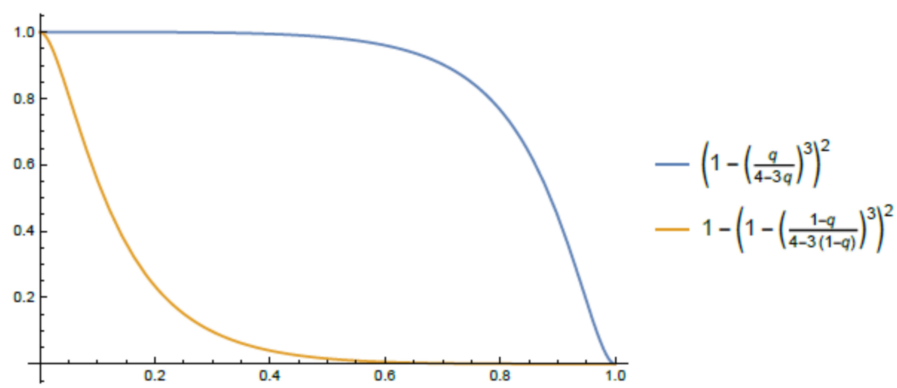
$$S_{P-S}(q) = 1 - \left[1 - \frac{(\frac{\theta(1-q)}{1-\theta(1-q)})^k}{(\frac{\theta}{1-\theta})^k}\right]^M = 1 - \left[1 - \frac{[(1-q)(1-\theta)]^k}{(1-\theta(1-q))^k}\right]^M.$$

In contrast to the Example 2, the graphical representations below of $S_{S-P}(q)$ and $S_{P-S}(q)$ show us that $S_{S-P}(q) > S_{P-S}(q)$ for each $q \in [0,1]$, i.e., the comparison of the reliability of networks $A$ and $B$ is predetermined ,in the sense that type $A$ networks are more reliable than type $B$ networks as soon as the conditions of the above consequence are met.

*a*) for $M = 2$, $k = 3$, $p = 1/4$, $\theta = 3/4$ we have that the functions

$$S_{S-P}(q) = \left[1 - \left(\frac{q}{4-3q}\right)^3\right]^2, \ S_{P-S}(q) = 1 - \left[1 - \left(\frac{1-q}{4-3(1-q)}\right)^3\right]^2$$

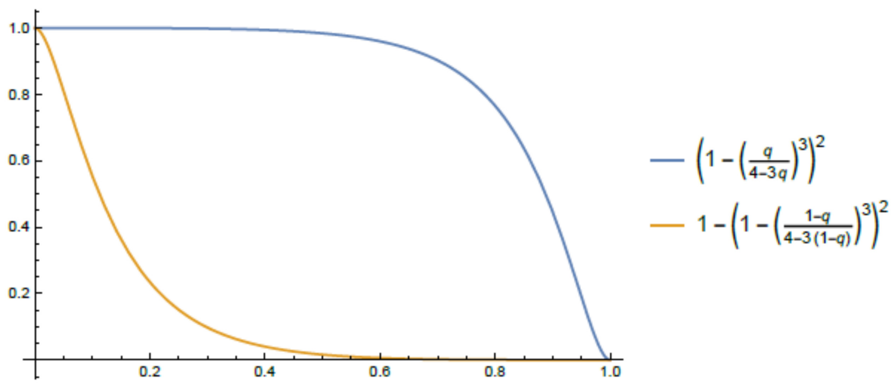
and their graphical representation is the following:




*b*) for $M = 2$, $k = 3$, $p = 3/4$, $\theta = 1/4$ we have that the functions

$$S_{S-P}(q) = \left[1 - \left(\frac{3q}{4-q}\right)^3\right]^2, \ S_{P-S}(q) = 1 - \left[1 - \left(\frac{3(1-q)}{4-(1-q)}\right)^3\right]^2$$

and their graphical representation is the following:




Finally we can draw the following

**Conclusions.** For serial-parallel and parallel-serial networks with constant number of subnetworks and random number of units in each subnetwork:

1. In the case when lifetimes of network's units are i.i.d.r.v., but also if the number of units in each sub-net is a r.v. which belong to the PSD class of r.v., the number of sub-networks being bigger than one, it was proved that the lifetime of the serial-parallel and parallel-serial networks belong to the class of Min(Max-PSD) and Max(Min-PSD) and may be calculated by means of the formulas (1)-(2), respectively.

2. In the same case, the reliability of the serial-parallel and parallel-serial networks may be calculated by formulas (3)-(4) respectively.

3. This formulas shows that solving the problem of identifying the best network in terms of its reliability the lifetime c.d.f. $F(x)$ of each unit in each subnetwork does not matter, the answer depending only of the number $M$ and the probability distribution of the number of units in each of $M$ sub-networks.

4. Sufficient conditions have been formulated for the serial-parallel network to always be more reliable than the parallel-serial network. Some examples have been illustrated graphically.

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